Super-Robust Endogenous Growth: Theory and Estimation.

Filippo Massari[*] Pietro F. Peretto[†]

Abstract

We propose an endogenous growth model that accommodates increasing, constant, or decreasing aggregate returns to scale with respect to the growth driving factor: quality-improving knowledge accumulated by firms in house. When aggregate production is non-linear in firm knowledge, the profitability of firms reflects that property, and entry (new product creation) responds accordingly. The consequent changes in market share offset the non-constant aggregate returns to scale and deliver constant firm-level returns to innovation in steady state. Because returns to innovation are constant, the steady-state growth rate of income per capita is constant and fully endogenous (i.e., dependent on policy parameters). The non-linearity with respect to the growth driving factor has testable implications for convergence dynamics. Specifically, the speed of convergence is decreasing (increasing) in the distance from the steady state when aggregate returns to firm knowledge are increasing (decreasing), causing asymmetric convergence dynamics. This propagation mechanism ensures that, subject to symmetric shocks, the average growth rate across shocks differs from the steady state rate, implying that these shocks' frequency and magnitude are a determinant of long-run growth. The model reduces the growth dynamics to a single quadratic differential equation in the growth rate of GDP per capita. The quadratic term captures the non-linearity in convergence and uniquely identifies the aggregate returns to firm knowledge. We estimate this equation on a panel of countries in the post-industrial revolution era finding evidence of increasing aggregate returns to firm knowledge.

1 Introduction

The conventional understanding in modern growth theory is that in lab-equipment models aggregate output must be a linear function of the growth driving factor(s) to deliver a constant, scale-invariant, endogenous growth rate of output per capita. In this paper, we challenge this understanding by highlighting that entry of new firms creates a competitive pressure on profits that preserves constant firm-level returns to investment even when aggregate production is non-linear in the growth driving factor(s). We then develop a simple empirical representation that relies exclusively on income per capita data to estimate the key parameter of the aggregate production function.

[*]Department of Economics, Dolan School of Business, Fairfield University. E-mail: fmassari@fairfield.edu.
[†]Department of Economics, Duke University. E-mail: peretto@econ.duke.edu.

Our argument relies on an endogenous growth model in which the growth-driving factor is quality-improving knowledge accumulated by firms in house. Henceforth, we refer to this first model ingredient as "firm knowledge" for brevity.[1] The second model ingredient is profit-driven entry of new firms that bring to the market new goods in competition with the existing ones.[2] The model's core mechanism is thus simple: existing firms fight for the market share through incremental quality innovation; new firms develop new products to claim their own initial market share, thereafter joining the incremental fight for the market share as newly established incumbents that do their own quality innovation. For the purposes of this paper, the key property of this mechanism is that it allows for constant steady-state growth with increasing, constant or decreasing aggregate returns to firm knowledge.

When incumbent firms accumulate knowledge in house to increase product quality, their operating profits increase more or less than proportionally to the increase in their knowledge depending on whether the aggregate production function is convex or concave in (average) firm knowledge. With a single dimension of technology, this non-linearity would cause the return to firm investment to either rise or decline over time. However, the free entry condition ensures that the return to new product creation mirrors the return to firm investment because it is driven by the operating profits that entrants anticipate they will make. Hence, when the aggregate production function is convex (concave) in firm knowledge, profits rise more (less) than proportionally with firm knowledge, leading to a more (less) than proportionate increase in the number of firms, thus diluting (concentrating) the market share of incumbents. This market share dilution/concentration mechanism counters the curvature of aggregate production with respect to firm knowledge and delivers a constant rate of return to firm investment in steady state. The result is a constant steady-state firm investment rate and a constant steady-state economic growth rate regardless of whether aggregate production is convex, linear, or concave in firm knowledge.

Our key theoretical result, therefore, is that endogenous growth is a robust outcome: it occurs for any positive value of the parameter measuring the aggregate returns to firm knowledge. Moreover, whether aggregate production is convex, linear, or concave in firm knowledge has important qualitative implications for the dynamics of the growth rate that are testable. When the aggregate returns to firm knowledge are not constant, the model's convergence dynamics are non-linear and the speed of convergence is asymmetric. The direction of this asymmetry depends exclusively on the parameter that determines the aggregate returns to firm knowledge. In particular, when growth is above its steady state value, the model converges faster than when growth is below its steady state value if aggregate returns to firm knowledge are decreasing. In the case of aggregate increasing returns, instead, convergence is faster from below.

This property has a novel implication. Due to the scale invariance property, the model features parameters that affect the detrended steady-state level of income per capita (level effect) but do not affect the steady-state growth rate of income per capita. Such parameters are those that affect the scale of the economy, like population and the land endowment. In particular, there is a specific combination of the population and land that regulates the scale of the economy. We refer to this combination as the scale factor. We then imagine

[1] Many contributions call the accumulation of quality-improving knowledge vertical innovation.

[2] This activity is typically called horizontal or variety-expanding innovation.

two unforeseen and equally probable changes in the scale factor of opposite sign. That is, we construct a symmetric shock that with equal probability displaces the economy to the right or to the left of the steady state.[3] The idea is to generate two possible transitions that occur with equal probability and start at the same distance from the steady state. We then compute the average growth rate across the two transition paths at any point in time and compare it to the steady state growth rate, which by construction is invariant to the two shocks. We find that under constant aggregate returns to firm knowledge, the average growth rate equals the steady state growth rate at any point in time. That is, the two symmetric shocks offset each other.[4] When aggregate returns to firm knowledge are not constant, instead, the offset does not occur.

The asymmetric convergence dynamics imply that convergence from above and from below occur at different speeds. Therefore, the growth rate spends more time away from its steady-state value in one case relative to the other. As a result, throughout the transition back to the steady state the across-shocks average growth rate is below the steady-state growth rate when aggregate returns to firm knowledge are decreasing. The reverse holds when aggregate returns to firm knowledge are increasing. This property says that shocks to the scale of the economy that do not affect the (deterministic) steady-state growth rate are nevertheless a potential determinant of the growth rate calculated as the average over a period of time, as in the standard empirical definition of long-run growth. The reason is that they displace systematically the economy from the deterministic steady state and thus initiate a propagation process that is asymmetric. Therefore, even if the shocks are symmetric in terms of size and persistence, the economy responds asymmetrically, spending more time on one or the other side of the steady state. This results in a measured growth rate that differs from the deterministic growth rate.

The model reduces the growth dynamics to a quadratic differential equation in the growth rate of income per capita, which belongs to the Riccati class. This mathematical structure has important theoretical and empirical implications. On the theoretical front, the Riccati equation has an explicit analytical solution that allows for easy computation of the full growth dynamics subject to any initial condition. On the empirical side, the coefficient of the quadratic term governs whether the speed of convergence to steady-state growth increases, decreases, or stays constant as growth deviates from its steady-state value. Moreover, this coefficient depends uniquely on the parameter determining aggregate returns to firm knowledge. Therefore, we can run a simple regression based exclusively on income per capita data and identify precisely this key parameter of the model.

Specifically, we transform the differential equation describing the convergence process into a quadratic regression equation that we run on a panel of 21 countries for the post Industrial Revolution period. We find that the data favor a model with increasing aggregate returns to firm knowledge. Moreover, if we drop the quadratic term and focus on the nested linear representation, we obtain a linearized speed of convergence of 2% per year, in line with the common results from the conditional convergence literature. Finally, we use our regression equation to compute the steady-state growth rate from the estimates of the coefficients and

[3]This is the simplest type of shock that we can consider. However, the result generalizes to any other type of shock that does not alter the steady state growth rate.

[4]This property suggests that if we were to write a stochastic version of the model with symmetric shocks, we would obtain that the average growth rate equals the deterministic growth rate.

obtain realistic values for most of the countries in our sample.

Robustness has always been a main point of contention concerning endogenous growth. Stiglitz (1990) and Romer (1994) argued that the unwillingness to work with knife-edge assumptions on the engine-of-growth equations has prevented the development of endogenous growth models for decades, explaining the long gap between Solow (1956 and 1957) and Romer (1986 and 1990).

Several attempts had introduced key features of endogenous growth theory long before Romer, such as profit-driven private R&D investment for the purpose of accumulating knowledge, a non-rival and partially non-excludable factor, and monopolistic competition that allows for pricing above marginal costs to compensate R&D investment (see, among many others, Arrow 1962, Nordhaus 1969, Judd, 1985, Grossman and Helpman 1989). However, none of these attempts deployed the ingredient that later became recognized as the key to endogenous growth.

In his account of the origins of the theory, Romer (1994, p. 18) says that "Nordhaus and Arrow both worked at a time when there was real concern about the knife-edge character of the assumptions about ϕ", where ϕ is the parameter measuring aggregate returns to knowledge in the engine-of-growth equation of his 1990 model of knowledge accumulation. Likewise, he says about Judd (1985) and Grossman and Helpman (1989) that "like Nordhaus and Arrow, they stayed well away from the case where ϕ was equal to 1."

This standstill ended when Romer (1986) proposed his first model in which a sustained rate of economic growth originates from the endogenous accumulation of technological knowledge according to a linear equation. Unsatisfied that the model could not handle profit-driven knowledge accumulation by firms, Romer developed the 1990 model, which shares the same linear mathematical representation of knowledge accumulation. Given the preference for simple models that produce a constant growth rate of income per capita, in those early days the theory of endogenous growth developed around the knife-edge assumption of constant returns to knowledge in knowledge accumulation, with the understanding that robustness was desirable but an unnecessary mathematical complication. In his recent and exhaustive review of the field, Bond-Smith (2019) argues that the tradeoff between getting rid of the knife-edge assumption and delivering constant growth rates lasted until Peretto's development of a model of robust endogenous growth (2018), which showed that in a lab-equipment model of the class that we study in this paper, an endogenous and constant growth rate of income per capita can emerge even in the presence of aggregate increasing returns to knowledge.

This paper builds on this tradition and makes four main contributions. First, it goes beyond Peretto (2018) by showing that it is possible to obtain the same results even with decreasing aggregate returns to knowledge, thus overturning the conventional wisdom in growth theory. In this sense, endogenous growth theory becomes super-robust. Second, we illustrate the importance for out-of-steady-state dynamics of the parameter that determines the aggregate returns to firm knowledge. This parameter determines the direction of the asymmetry in the speed of convergence and is thus identified cleanly by the data. Therefore, third, we exploit this property to estimate the aggregate returns to firm knowledge from the data.[5] Moreover, fourth, we estimate the steady-state growth rate from the regression

[5] An example of the type of application that is possible within this framework is Chu *et al.* (2005). Whereas

equation that we obtain from the model's transitional dynamics instead of imposing it as an external estimate as in the conditional convergence regressions tradition.[6]

Our empirical strategy focuses on the speed of convergence, a topic reviewed by Durlauf *et al.* (2005) and more recently by Johnson and Papageorgiou (2020). Our approach is quite different from this tradition. First, due to the nature of our work, we are concerned exclusively with countries that grow predominantly through expanding the knowledge frontier. Therefore, we exclude emerging or low-income economies where the theory that we propose is less relevant. Exogenous variation is, therefore, not in the initial income levels but in exogenous changes to their initial growth rate of income, after which the growth rate is expected to converge to its steady-state value. Second, although the speed of convergence that we obtain when we use a linear approximation of our quadratic equation aligns with the literature, our focus is primarily on the non-linearity of our equation because our theoretical model says that convergence accelerates or decelerates due to specific properties of the knowledge accumulation process that the literature has so far ignored.

We organize the paper as follows. Section 2 sets up the model. Section 3 contructs the model's equilibrium and derives the representation of the dynamics. Section 4 derives and discusses our robustness result. Section 5 constructs the mapping between model and data. Section 6 performs the estimation and discusses the paper's empirical result. Section 7 concludes.

2 Model

A representative household earns income from financial assets, labor and land and makes consumption/saving and labor/leisure choices. The production sector consists of a representative competitive firm that assembles a final good from a set of differentiated intermediate goods, each supplied by a monopolist that accumulates quality-improving knowledge in house (vertical innovation). Entrants create new intermediate goods (horizontal innovation) to capture a share of the intermediate goods market according to a free-entry condition.

2.1 Household

The representative household has lifetime utility function

$$U = \int_0^\infty e^{-\rho t} \Lambda(t) \left[\ln c(t) + \eta \ln (1 - l(t)) \right] dt, \quad \rho > \lambda > 0, \ \eta \geq 0 \tag{1}$$

where the parameter ρ is the subjective discount rate and $\Lambda(t) = \Lambda_0 e^{\lambda t}$ is the the mass of identical household members (population) which grows at the exogenous rate λ. The variable

this paper focuses on the theoretical implications of non-constant returns to knowledge accumulation and estimates such returns, that paper studies the relevance of government expenditure for the timing of the industrial takeoff and the subsequent evolution of economic growth.

[6] In conditional convergence regressions, the steady-state growth rate is typically calculated as the slope of the linear representation of the log of GPD per capita. This is valid for some countries (e.g., USA, UK) but not for all. Moreover, the practice is to impose that the steady-state growth rate is the same for all countries.

c denotes consumption per capita of a final good (our numeraire good) defined in the next subsection. Aggregate consumption is $C \equiv c\Lambda$. Finally, the variable $l \in (0,1]$ is the fraction of time that each household member allocates to work and the parameter η determines the importance of leisure relative to consumption.

The household maximizes utility subject to the asset-accumulation equation

$$\dot{A} = rA + w_L L + w_\Omega \Omega - C, \tag{2}$$

where A is financial wealth , r is the real interest rate, w_L is the real wage and w_Ω is the real price of land services, which the household supplies inelastically from the fixed endowment Ω. Dynamic optimization yields the Euler equation

$$r = \rho - \lambda + \frac{\dot{C}}{C} \tag{3}$$

and the individual labor supply curve

$$l = 1 - \frac{\eta C/\Lambda}{w_L}. \tag{4}$$

2.2 Final good

A representative competitive firm produces the final good with the technology

$$Y = \int_0^N X_i^\theta \left(Z_i^\alpha Z^{\kappa-\alpha} L_i^\gamma \Omega^{1-\gamma} \right)^{1-\theta} di, \quad \theta, \gamma \in (0,1)\,,\ \alpha > 0,\ \kappa > 0 \tag{5}$$

where N is the mass of differentiated intermediate goods that the economy knows how to produce at a point in time, X_i is the quantity of intermediate good i, Z_i is the knowledge of firm i and $Z \equiv \int_0^N (Z_j/N)\, dj$ is average firm knowledge. We interpret this Cobb-Douglas production structure as featuring constant returns to scale (CRS) with respect to two rival inputs, intermediate goods and a knowledge-augmented composite of labor and land. The augmentation term is $Z_i^\alpha Z^{\kappa-\alpha}$.

This structure treats labor as rival with respect to intermediate goods, that is, the labor L_i assigned to intermediate i cannot simultaneously work with intermediate $j \neq i$. The structure also treats the land input Ω as non-rival across intermediate goods and not subject to congestion. Consequently, the technology features social returns to variety of degree $1-\gamma$ (more on this below). Moreover, as shown in previous work, this structure delivers endogenous growth with $\kappa = 1$, a restriction that reduces the model to the one first presented in Peretto (2015) and further elaborated in Peretto (2018).

Differently from those previous contributions, in this paper we do not impose a priori restrictions on the parameters designed to produce endogenous growth driven by the accumulation of knowledge by firms but, instead, we derive such restrictions from the equilibrium of the model under two criteria. First, the model must produce a steady state with endogenous growth defined as in Romer (1986, 1990), that is, a constant growth rate of income per capita that is (a) generated within the model and (b) subject to policy action. Second, the model must deliver a transition to the steady state with endogenous growth. Our main theoretical result is that the model admits $\kappa \lesseqgtr 1$ because it generates endogenous growth as defined by Romer regardless of whether in equilibrium the reduced-form aggregate production function is concave, linear or convex in average firm knowledge Z, which is the model's growth driving factor.

To close this subsection, we now characterize the behavior of the representative final producer. Profit maximization yields the conditional demands for labor and land,

$$L = \gamma (1-\theta) \frac{Y}{w_L} \quad \textit{and} \quad \Omega = (1-\gamma)(1-\theta) \frac{\Omega}{w_\Omega}, \tag{6}$$

and the conditional demand for each intermediate good,

$$X_i = \left(\frac{\theta}{p_i}\right)^{\frac{1}{1-\theta}} Z_i^\alpha Z^{\kappa-\alpha} L_i^\gamma \Omega^{1-\gamma}, \tag{7}$$

where p_i is the price of good i. Accordingly, the final producer pays $w_L L = \gamma (1-\theta) Y$ for labor, $w_\Omega \Omega = (1-\gamma)(1-\theta) Y$ for land and $\theta Y = \int_0^{N_t} p_i X_i di$ for intermediate goods.

2.3 Intermediate goods

A monopolistic firm produces intermediate good i with a linear technology that uses X_i units of final good to produce X_i units of intermediate good i at quality Z_i. This implies that the marginal cost of production is one. The firm also pays ϕZ units of final good as a fixed operating cost.[7] Finally, to improve the quality of its product, the firm allocates I_i units of final good to in-house R&D with the innovation technology

$$\dot{Z}_i = \zeta I_i, \quad \zeta > 0. \tag{8}$$

The firm's gross profit is

$$\Pi_i = (p_i - 1) X_i - \phi Z = (p_i - 1) \left(\frac{\theta}{p_i}\right)^{\frac{1}{1-\theta}} Z_i^\alpha Z^{\kappa-\alpha} L_i^\gamma \Omega^{1-\gamma} - \phi Z \tag{9}$$

and the value of the firm is

$$V_i(t) = \int_t^\infty e^{-\int_t^s r_u du} \left[\Pi_i(s) - I_i(s)\right] ds. \tag{10}$$

The firm maximizes (10) subject to (7) and (8).

This firm-level problem is well defined if and only if it features concavity with respect to the firm-specific state variable Z_i, which requires $0 < \alpha < 1$. We stress that this restriction has nothing to do with the ability of the model to generate endogenous growth but is due to the model's deeper micro structure, in particular, the requirement that the investment problem of the typical intermediate firm be well-defined in the sense that the individual

[7] Our results are robust to a more general specification for the fixed operating cost: $\phi Z_t^\chi(i) Z_t^{1-\chi}$, where $\chi \in (0,1)$. The specification in the text with $\chi = 0$ is the simplest one.

firm cannot take over the entire intermediate goods market exploiting firm-level dynamic increasing returns to scale that are too strong.

Dynamic optimization of the monopolistic firm yields the unconstrained profit-maximizing markup ratio $1/\theta$. However, we follow Chu *et al.* (2020) to allow for diffusion of knowledge from monopolistic firms to competitive fringe firms, which can produce X_i with the same quality Z_i but at the higher marginal cost $\mu > 1$.[8] To price these fringe firms out of the market, the monopolistic firm sets

$$p_i = \min\left\{\mu, 1/\theta\right\} = \mu, \tag{11}$$

where we assume $\mu < 1/\theta$. The firm's optimization problem also yields the rate of return to in-house R&D

$$r_i^Z = \alpha\zeta\left(\mu - 1\right)\frac{X_i}{Z_i},$$

which is proportional to quality-adjusted firm size X_i/Z_i.

The process of firm creation is as follows. A new firm pays βX units of the final good, where $\beta > 0$ is an entry-cost parameter and $X \equiv \int_0^N \left(X_j/N\right) dj$ is average firm size, to develop a new differentiated good with average quality Z and start serving the market.[9] Once in the market, the new firm behaves like the typical incumbent characterized above. Therefore, at any point in time the value of all firms — incumbents and entrants — is governed by the asset-pricing equation

$$r_i^N = \frac{\Pi_i - I_i}{V_i} + \frac{\dot{V}_i}{V_i}. \tag{12}$$

When entry is positive, the free-entry condition $V_i = \beta X$ holds.

As shown in previous contributions (e.g., Peretto 2015), the industry equilibrium of this model is symmetric, i.e., firms set identical prices and grow at the same rate. Therefore, henceforth we can drop the firm subscript i and interpret firm-level variables as industry averages. Accordingly, substituting (7), (8), (9), (11) and $V_i = \beta X$ into (12) yields the return to entry as

$$r^N = \frac{1}{\beta}\left(\mu - 1 - \frac{\phi + z}{X/Z}\right) + \frac{\dot{X}}{X},$$

where $z \equiv \dot{Z}/Z$ is the average growth rate of quality, which is also the growth rate of average quality. The return to entry r^N, equivalently, the return to equity holding, is increasing in quality-adjusted firm size X/Z.

[8] This characterization is not necessary to derive the model's theoretical properties but is convenient for its calibration because it disentangles the markup from the parameter θ.

[9] This characterization of entry preserves the symmetry of the intermediate goods market equilibrium at all times. Furthermore, the dependence of the entry cost on X ensures that the steady state entry rate is trendless. The reason is that production costs increase linearly in X as well. In a growing economy the entry cost must scale with the production cost to ensure that the ratio of resources devoted to the two activities remains constant. See Klenow and Li (2024) for recent empirical evidence showing that the entry costs scale with the level of economic development, which supports our assumption.

3 Equilibrium dynamics

In this section we construct the equilibrium dynamics of the model in the three different cases of decreasing, constant, or increasing returns to the growth driving factor, Z. To do so, we define a composite variable that serves as the fundamental state variable of the model and show that the global dynamics of the model reduces to a single differential equation in this variable.

3.1 Equilibrium

The equilibrium of this economy is a time path of allocations $\{A, C, L, Y, X, I\}$ and prices $\{r, w_L, w_\Omega, p, V\}$ such that:

- the household chooses consumption and labor supply to maximize utility taking prices as given;
- the competitive final producers maximize profits taking prices as given;
- the monopolistic intermediate firms choose $\{p, I\}$ to maximize V taking r as given;
- the entrants make decisions taking the maximized value V as given;
- the aggregate value of monopolistic firms equals the household's wealth, $A = NV$;
- the labor market clears;
- the market for the final good clears, $Y = C + G + N(X + \phi Z + I) + \dot{N}\beta X$.

3.2 Output, firm size and growth

In our symmetric equilibrium, equations (7) and (11) yield the reduced-form aggregate production function

$$Y = \left(\frac{\theta}{\mu}\right)^{\frac{\theta}{1-\theta}} N^{1-\gamma} Z^{\kappa} L^{\gamma} \Omega^{1-\gamma}, \tag{13}$$

where the elasticity of output with respect to firm knowledge, Z, is κ. Note that by construction the parameter $\alpha < 1$ that regulates the private returns to firm knowledge drops out of this representation, leaving the unrestricted parameter κ, that regulates the aggregate or social returns to firm knowledge, to be the sole determinant of the accumulation dynamics.[10] Recall that the final producer spends $NpX = \theta Y$ on intermediate goods. Therefore, GDP in this economy is

$$G = Y - NX - N\phi Z = \left[1 - \frac{\theta}{\mu}\left(1 + \frac{\phi}{X/Z}\right)\right] Y. \tag{14}$$

[10] Specifications that allow α to affect the dynamics are feasible and do not change the paper's results. We prefer the specification in the text because it cleanly and unambiguously identifies the social returns to firm knowledge — and aggregate rather than a firm-level concept — as the key to the model's dynamics.

Using (7) and (11), we express quality-adjusted firm size as

$$\frac{X}{Z}=\left(\frac{\theta}{\mu}\right)^{\frac{1}{1-\theta}}Z^{\kappa-1}\left(\frac{L}{N}\right)^{\gamma}\Omega^{1-\gamma}=\varpi\frac{Z^{\kappa-1}\Lambda^{\gamma}\Omega^{1-\gamma}}{N^{\gamma}}=\varpi x,$$

where for notational convenience we define the composite parameter

$$\varpi\equiv\left(\frac{\theta}{\mu}\right)^{\frac{1}{1-\theta}}\left[1+\eta+\frac{\eta\beta\theta(\rho-\lambda)}{\mu(1-\theta)}\right]^{-\gamma}, \tag{15}$$

and the composite state variable

$$x\equiv\frac{Z^{\kappa-1}\Lambda^{\gamma}\Omega^{1-\gamma}}{N^{\gamma}}. \tag{16}$$

This composite state variable compresses the three state variables Λ (population), Z (average quality) and N (mass of products/firms) and therefore makes the analysis of the model's dynamics remarkably simple.

For the purposes of this paper, we focus on the model's equilibrium when both horizontal and vertical innovation are active. When the free-entry condition holds, the consumption-output ratio C/Y jumps to its steady-state value[11]

$$\left(\frac{C}{Y}\right)^{*}=1-\theta+\frac{\beta\theta}{\mu}(\rho-\lambda). \tag{17}$$

The combination of labor supply (4) and labor demand (6) then yields the equilibrium fraction of time allocated to work

$$l^{*}=\frac{1}{1+\frac{\eta}{1-\theta}\frac{C}{Y}}=\frac{1}{1+\eta\left[1+\frac{\beta\theta(\rho-\lambda)}{\mu(1-\theta)}\right]}. \tag{18}$$

The growth rate of final output then is

$$\frac{\dot{Y}}{Y}=\gamma\lambda+(1-\gamma)\,n+\kappa z. \tag{19}$$

This growth rate has two components: the growth rate of product variety, $n\equiv\dot{N}/N$, and growth rate of product quality, $z\equiv\dot{Z}/Z$. The associated growth rate of GDP per capita is

$$g\equiv\frac{\dot{G}}{G}-\lambda=\frac{\dot{Y}}{Y}-\lambda+\frac{\frac{\theta}{\mu}\frac{\phi}{\varpi x}}{1-\frac{\theta}{\mu}\left(1+\frac{\phi}{\varpi x}\right)}\frac{\dot{x}}{x}. \tag{20}$$

The last term in this expression describes the contribution to GDP growth of the static economies of scale due to the fixed operating cost, the term in brackets in equation (14).

[11] See Peretto (2015) for the case $\kappa=1$. The proof is identical for the other two cases that we study in this paper (see, e.g., Peretto 2018 for the case $\kappa>1$).

3.3 Rates of return and equilibrium law of motion

We now use equation (16) to write the rate of return to firm innovation as

$$r = \alpha\zeta(\mu - 1)\varpi x \tag{21}$$

and the rate of return to entry as

$$r = \frac{1}{\beta}\left[\mu - 1 - \frac{\phi + z/\zeta}{\varpi x}\right] + \frac{\dot{Y}}{Y} - n. \tag{22}$$

Both rates of return are increasing in quality-adjusted firm size $X/Z = \varpi x$ and are thus decreasing in the mass of firms. This property captures the main force driving this class of models: as the mass of firms rises, each firm captures a smaller share of the market and experiences falling profitability and thereby a weaker incentive to innovate.

These two expressions are important for our purposes. As the returns to vertical and horizontal innovation are functions of only one variable, x, proving that the returns to innovation are constant requires proving the existence of a steady state for the variable x. To do it, we proceed as follows. Equation (16) yields the equilibrium law of motion

$$\frac{\dot{x}}{x} = \gamma\lambda + (\kappa - 1)\, z - \gamma n. \tag{23}$$

In this expression, the entry rate n and the quality growth rate z are increasing functions of quality-adjusted firm size ϖx (as we show below). The core of the analysis in this section is the characterization of these two functions as equilibrium objects.

Since C/Y is constant, the Euler equation becomes $r = \rho - \lambda + \dot{Y}/Y$. We combine this result with the returns to firm innovation and to firm entry to write:

$$\frac{\dot{Y}}{Y} = \alpha\zeta(\mu - 1)\varpi x - \rho + \lambda; \tag{24}$$

$$n = \frac{1}{\beta\varpi x}\left[(\mu - 1)\,\varpi x - \phi - \frac{z}{\zeta}\right] - \rho + \lambda. \tag{25}$$

Using these two expressions and equation (19), a little bit of algebra yields the growth rate of quality z as a function of the state variable x, namely,

$$z\,(x) = \frac{\alpha\zeta\,(\mu - 1)\,\varpi x - \frac{1-\gamma}{\beta}\,(\mu - 1) - \gamma\rho + \phi\frac{1-\gamma}{\beta\varpi x}}{\kappa - \frac{1-\gamma}{\beta\varpi x}\frac{1}{\zeta}}. \tag{26}$$

This expression says that quality growth is positive if and only if $x > x_Z$, where

$$x_Z \equiv \arg\underset{x}{\text{solve}}\left\{\alpha\zeta\,(\mu - 1)\,\varpi x + \phi\frac{1-\gamma}{\beta\varpi x} = \frac{1-\gamma}{\beta}\,(\mu - 1) + \gamma\rho\right\}. \tag{27}$$

Substituting (26) in (25) and rearranging terms, we write

$$n\,(x) = \frac{(\mu - 1)\,\varpi x - \phi}{\beta\varpi x} - \frac{1}{\zeta}\frac{\alpha\zeta\,(\mu - 1)\,\varpi x - \frac{1-\gamma}{\beta}\,(\mu - 1) - \gamma\rho + \phi\frac{1-\gamma}{\beta\varpi x}}{\kappa\beta\varpi x - \frac{1-\gamma}{\zeta}} - \rho + \lambda, \tag{28}$$

which expresses the rate of entry as a function of the state variable x. This equation as well identifies a threshold

$$x_N \equiv \arg\underset{x}{\text{solve}}\left\{\frac{(\mu-1)\varpi x-\phi}{\beta\varpi x}-\frac{1}{\zeta}\frac{\alpha\zeta(\mu-1)\varpi x-\frac{1-\gamma}{\beta}(\mu-1)-\gamma\rho+\phi\frac{1-\gamma}{\beta\varpi x}}{\kappa\beta\varpi x-\frac{1-\gamma}{\zeta}}=\rho\right\},$$

such that investment in variety growth is zero for $x \leq x_N$. Following the literature, we work with parameter values such that $x > x_N$.

The model's global dynamics are well-defined for $(\mu-1)\varpi x > \phi$, which yields positive firm profit, and thus one can use the model to study topics like the transition from stagnation to sustained growth (see, e.g., Peretto 2015). For the purpose of this paper, however, restricting our attention to the region $x > x_Z$ is sufficient since we are interested in the model's behavior when both n and z are positive. We thus use equations (26) and (28) to write the equilibrium law of motion

$$\dot{x}=(\kappa-1)\,z(x)\,x+\frac{\gamma}{\beta\varpi}\frac{z(x)}{\zeta}+\gamma\rho x-\frac{\gamma}{\beta}(\mu-1)\,x+\frac{\gamma\phi}{\beta\varpi}.$$

After a bit of algebra, the expression reduces to

$$\dot{x}=\frac{\omega_2 x^2-\omega_1 x+\omega_0}{\kappa-\frac{1-\gamma}{\beta\varpi\zeta}\frac{1}{x}}, \tag{29}$$

where:

$$\omega_2 \equiv (\kappa-1)\,\alpha\zeta(\mu-1)\varpi; \tag{30}$$

$$\omega_1 \equiv (\mu-1)\frac{1}{\beta}\left[\kappa-1+\gamma(1-\alpha)\right]-\gamma\rho > 0; \tag{31}$$

$$\omega_0 \equiv \left[(\kappa-1+\gamma)\,\phi-\frac{\gamma\rho}{\zeta}\right]\frac{1}{\beta\varpi} > 0. \tag{32}$$

This is an extremely parsimonious expression. The numerator is a quadratic form. The denominator is positive for x sufficiently large, i.e., for $x > x_Z > \frac{1-\gamma}{\beta\varpi\kappa\zeta}$. Therefore, establishing that the model delivers endogenous growth with decreasing, constant, or increasing aggregate returns to firm knowledge simply requires showing that equation (29) has a unique stable steady state.

3.4 The steady state

The steady state that interests us is the solution

$$x^*=\frac{\omega_1-\sqrt{\omega_1^2-4\omega_2\omega_0}}{2\omega_2}; \tag{33}$$

of the quadratic equation

$$\omega_2 x^2-\omega_1 x+\omega_0=0. \tag{34}$$

We then obtain the steady-state rates of vertical and horizontal innovation:

$$z^* = \frac{\alpha\zeta(\mu-1)\varpi x^* - \frac{1-\gamma}{\beta}(\mu-1) - \gamma\rho + \phi\frac{1-\gamma}{\beta\varpi x^*}}{\kappa - \frac{1-\gamma}{\beta\varpi x^*}\frac{1}{\zeta}}; \tag{35}$$

$$n^* = \frac{(\mu-1)\varpi x^* - \phi}{\beta\varpi x^*} - \frac{1}{\zeta}\frac{\alpha\zeta(\mu-1)\varpi x^* - \frac{1-\gamma}{\beta}(\mu-1) - \gamma\rho + \phi\frac{1-\gamma}{\beta\varpi x^*}}{\kappa\beta\varpi x^* - \frac{1-\gamma}{\zeta}} - \rho + \lambda. \tag{36}$$

The associated growth rate of GDP per capita is

$$g^* = \alpha\zeta(\mu-1)\varpi x^* - \rho, \tag{37}$$

which we can also write

$$g^* = \gamma\lambda + (1-\gamma)n^* + \kappa z^* - \lambda = (1-\gamma)(n^* - \lambda) + \kappa z^*$$

to emphasize the different contributions of the two innovation rates.

4 Super-Robust Endogenous Growth (SREG)

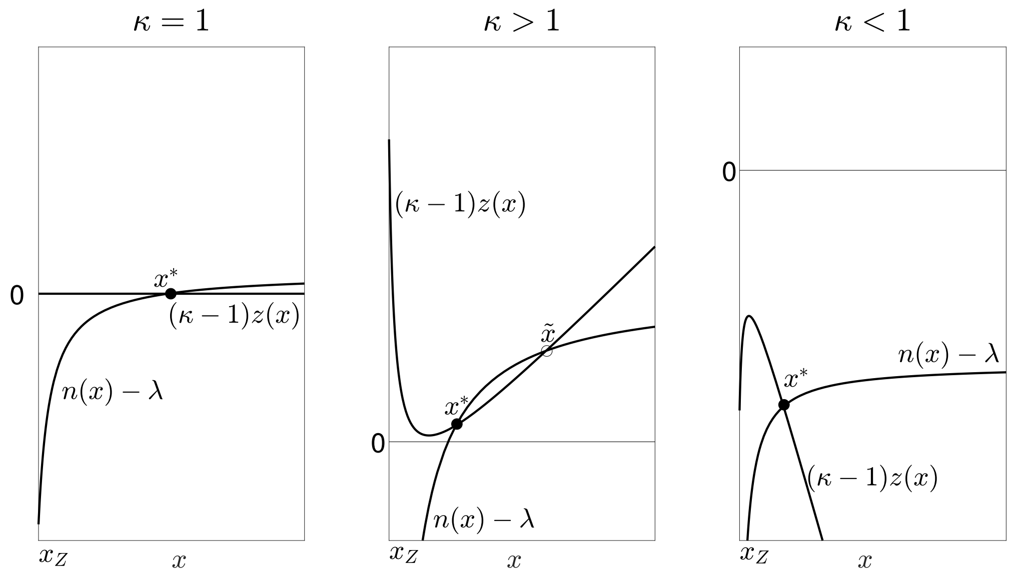
In this section we derive the paper's main theoretical result, namely, the model produces constant endogenous growth independently of whether the reduced-form production function (13) is concave, linear or convex in the engine-of-growth factor, Z. Figure 1 illustrates the three cases of constant, increasing, and decreasing aggregate returns to firm knowledge. The figure describes the combination of variety-expanding and quality-improving innovation rates that balances the various forces to deliver a steady state with constant endogenous growth. Specifically, the figure shows the solution of the equation

$$(\kappa-1)z(x) = \gamma[n(x) - \lambda].$$

In particular, one curve shows the growth rate of firm knowledge as the function $z(x)$ multiplied by the deviation of the aggregate return to knowledge κ from the canonical benchmark value of 1, while the other curve shows the growth rate of the number of firms per capita $n(x) - \lambda$ multiplied by the elasticity of final output with respect to labor γ. The economy is in steady-state when the two curves intersect. This happens at a growth rate of the number of firms per capita that is positive, zero, or negative depending on whether the returns to firm knowledge are increasing ($\kappa > 1$), constant ($\kappa = 1$), or decreasing ($\kappa < 1$). We state formally our first theoretical result in the following proposition.

Proposition 1 *(SREG) In each of the three cases $\kappa > 1$ (increasing returns to knowledge), $\kappa = 1$ (constant returns), and $\kappa < 1$ (decreasing returns), a unique stable steady state with constant endogenous growth exists under a thick set of parameter values. This set consists of the inequality conditions that guarantee that the solution x^* in equation (33) exists, is real and is positive plus the inequality condition that yields g^* positive in equation (37).*

Figure 1: Steady-State: Solution to $(\kappa - 1)\, z\,(x) = \gamma\,[n\,(x) - \lambda]$ in the Three Cases



Note: For simplicity we show only the graph for values of $x > x_Z$.

To prove this proposition, we need to characterize the combinations of parameters that deliver a unique endogenous-growth steady state that is stable under the model's dynamics. To establish stability, we use the phase diagrams in Figure 2 and note that it shows that the inequality

$$\dot{x}\,(x_Z) > 0 \Rightarrow \frac{\omega_2 x_Z^2 - \omega_1 x_Z + \omega_0}{\kappa - \frac{1-\gamma}{\beta\varpi\zeta}\frac{1}{x_Z}} > 0 \tag{38}$$

ensures that $\dot{x} > 0$ for all $x < x^*$. We call this restriction on the model's parameters the full transition condition. We proceed then case by case.

The canonical case $\kappa = 1$. Since $\omega_2 = 0$, we have

$$x^* = \frac{\omega_0}{\omega_1} = \frac{1}{\varpi}\frac{\phi - \frac{\rho}{\zeta}}{(\mu - 1)\,(1 - \alpha) - \rho\beta}. \tag{39}$$

The associated growth rate of income per capita is

$$g^* = \alpha\zeta(\mu - 1)\varpi x^* - \rho = (1 - \gamma)\,(n^* - \lambda) + z^* = z^*, \tag{40}$$

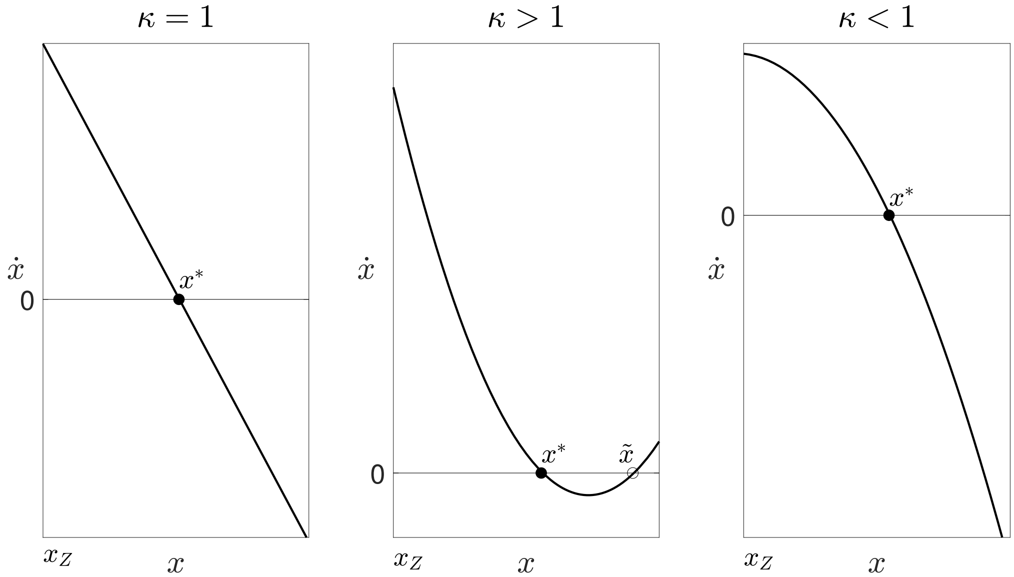
since $n^* = \lambda$. The conditions for endogenous growth then are: (i) the values x^*, g^* and z^* exist and are positive; (ii) under the model's dynamics the state variable x converges to the steady state x^* from any initial condition x_0. In this simple case, the latter condition is

$$\dot{x}\,(x_Z) > 0 \Rightarrow \frac{-\omega_1 x_Z + \omega_0}{\kappa - \frac{1-\gamma}{\beta\varpi\zeta}\frac{1}{x_Z}} = \frac{(x^* - x_Z)\,\omega_1}{\kappa - \frac{1-\gamma}{\beta\varpi\zeta}\frac{1}{x_Z}},$$

which reduces to

$$x^* > x_Z > \frac{1-\gamma}{\kappa\beta\varpi\zeta} \Rightarrow \frac{\phi - \frac{\rho}{\zeta}}{(\mu - 1)\,(1 - \alpha) - \rho\beta} > \varpi x_Z > \frac{1-\gamma}{\kappa\beta\zeta}. \tag{41}$$

Figure 2: Phase Diagram



Note: For simplicity we show only the phase diagram for values of $x > x_Z$.

The phase diagram then says that if the steady state x^* exists, it is stable and therefore the global attractor of the full dynamical system. The condition for $x^* > 0$ is

$$\frac{\phi - \frac{\rho}{\zeta}}{(\mu - 1)(1 - \alpha) - \rho\beta} > 0, \tag{42}$$

which requires that both the numerator and the denominator of (39) be positive. The conditions for $g^* = z^* > 0$ adds the inequality

$$\frac{\phi - \frac{\rho}{\zeta}}{(\mu - 1)(1 - \alpha) - \rho\beta} > \frac{\rho}{\alpha\zeta(\mu - 1)}, \tag{43}$$

which surely holds given the existence condition (42). Summarizing, the conditions that ensure that the economy converges to x^* under the model's equilibrium dynamics, where x^* exhibits endogenous growth, are the existence condition (42) and the full transition condition (41). Noting that the inequality (41) implies the inequality (42), we conclude that the full transition condition (41) is sufficient for our existence and stability result. We stress that all that we discussed here are *inequality* restrictions; to deliver endogenous growth, the model does not require an equality restriction on the parameters akin to the knife-edge condition discussed in the Introduction that characterizes the first-generation models of endogenous growth.

The general case $\kappa \neq 1$. The conditions for $x^* > 0$ are simply that x^* exists, is real and is positive, that is:

$$\omega_1^2 - 4\omega_2\omega_0 > 0;$$
$$\frac{\omega_1 - \sqrt{\omega_1^2 - 4\omega_2\omega_0}}{2\omega_2} > 0.$$

In this case as well these are inequality restrictions on the parameters. We then add the restrictions for $g^* > 0$ and $z^* > 0$ and the full transition condition (38). Figure 1 aids in seeing how these conditions relate to each other. It also shows that this general case exhibits the property that the steady-state entry rate, n^*, is increasing or decreasing in the growth rate of quality, z^*. In other words, quality innovation is either so effective that it creates room for variety growth faster than in the canonical case $\kappa = 1$, which yields $n^* = \lambda$, or so ineffective that it requires variety growth slower than in the canonical case. To understand why the model generates constant growth under seemingly explosive or implosive conditions, we revisit equation (16) that defines the composite state variable x. That equation identifies the ratio

$$\ell \equiv L/N$$

as the key measure of labor input per intermediate good. The interpretation is that this ratio measures the flow of raw labor services allocated to the typical intermediate good. In the steady state, this measure of labor allocation has growth rate

$$\frac{\dot{\ell}}{\ell} = \lambda - n^* = -\frac{\kappa - 1}{\gamma} z^*.$$

The interpretation of this steady state, therefore, is that the economy exhibits constant endogenous growth because the mass of firms grows sufficiently faster or slower than the population so that there is continuous dilution ($\kappa > 1$) or concentration ($\kappa < 1$) of labor services across firms. As illustrated in the second panel of Figure 1, the dilution of labor services offsets the explosive pressure due to the property that production is convex in average knowledge, Z. As illustrated in the third panel of Figure 1, in contrast, the concentration of labor services offsets the implosive pressure due to the property that production is concave in average knowledge, Z.

We can summarize the result of this section as follows. Endogenous growth, defined as a constant exponential rate of growth of income per capita that is fueled by profit-driven innovation and is subject to policy action, occurs in this model for any value of the elasticity of final output with respect to firm knowledge. In particular, the conditions for its occurrence in each of the three cases of final production concave ($\kappa < 1$), linear ($\kappa = 1$), or convex ($\kappa > 1$) in firm knowledge consist of a set of inequality restrictions on the parameters. Therefore, endogenous growth occurs for a thick set of parameter values and a thick set of initial conditions for quality-adjusted firm size that place the economy in the basin of attraction of the endogenous-growth steady state (x^*). The basin of attraction is the entire real line for the concave and linear cases (global stability), while it is the interval of the real line from the origin to the unstable steady state ($\tilde{x}$) in the convex case.

5 Convergence Dynamics: Taking the Model to the Data

We now map the state variable x defined in Section 4 into an observable variable that we can use in our empirical analysis. We then show that the speed of convergence is not constant and is asymmetric when the aggregate returns to vertical knowledge are not constant. Finally, we illustrate the implications for growth of the non-linear and asymmetric convergence dynamics.

5.1 A useful approximation

For the purpose of bringing the model to the data and discussing convergence dynamics without overly complicating the algebra, we use the approximation $1/x \approx 0$. The approximation implies $(1-\gamma)/\beta\varpi x \approx 0$ and thus yields the firm innovation rate

$$z(x) = \frac{\alpha\zeta(\mu-1)\varpi x - \frac{1-\gamma}{\beta}(\mu-1) - \gamma\rho}{\kappa}. \tag{44}$$

The key simplification is that this expression is linear in x and says that quality growth is positive if and only if $x > x_Z$, where

$$x_Z \equiv \frac{\frac{1-\gamma}{\beta}(\mu-1) + \gamma\rho}{\alpha\zeta(\mu-1)\varpi}. \tag{45}$$

The law of motion of x then reduces to the simple quadratic differential equation

$$\dot{x} = \varphi_2 x^2 - \varphi_1 x + \varphi_0 \tag{46}$$

where $\varphi_2 \equiv \omega_2/\kappa$, $\varphi_1 \equiv \omega_1/\kappa$, and $\varphi_0 \equiv \omega_0/\kappa$.

A remarkable property of this representation is that the differential equation that governs the dynamics of x is a Riccati equation that we solve explicitly (see the appendix for the full derivation). Specifically, let the two roots of the quadratic equation $\varphi_2 x^2 - \varphi_1 x + \varphi_0 = 0$ be x^* and $\tilde{x}$, with x^* being the stable root and $\tilde{x}$ being the unstable root. Given initial condition x_0, the equilibrium path of firm size is

$$x(t) = \frac{x^*(\tilde{x}-x_0) - \tilde{x}(x^*-x_0)e^{\varphi_2(x^*-\tilde{x})t}}{(\tilde{x}-x_0) - (x^*-x_0)e^{\varphi_2(x^*-\tilde{x})t}}. \tag{47}$$

This solution highlights that the coefficient φ_2 of the quadratic term in our differential equation governs the convergence dynamics. Moreover, it describes analytically all of the relevant properties of the dynamics that we wish to take to the data.

To this purpose, we can extract further insight by noting that in steady state $\varphi_0 = -\left(\varphi_2(x^*)^2 + \varphi_1 x^*\right)$. We can thus rewrite the differential equation in the form

$$\begin{aligned}\dot{x} &= \varphi_2 x^2 - \varphi_1 x - \varphi_2(x^*)^2 - \varphi_1 x^* \\ &= \left[\varphi_2(x-x^*) + 2\varphi_2 x^* - \varphi_1\right](x-x^*).\end{aligned}$$

This equation has steady state x^*, which is the solution of the equation $\varphi_2 x^2 - \varphi_1 x + \varphi_0 = 0$, which in turn is the solution of the equation $\omega_2 x^2 - \omega_1 x + \omega_0 = 0$ characterized in Section 4. This property says that the approximation that we use in this analysis is very mild and that the dynamics that we study here represent very well the original model. In particular, our procedure yields the same representation of the dynamics as taking a second order approximation of the $\dot{x}$ equation around the steady state. The reason is that equation (29) is the ratio of a quadratic form to the term $\kappa - \frac{1-\gamma}{\beta\varpi\zeta}\frac{1}{x}$, which converges very quickly to κ as x grows. It is worth stressing that (29) and (46) have the same steady state by construction.

5.2 Going to the data

While the analytical solution (47) is remarkable and useful, it has the problem that lacking data on firm knowledge, Z, the variable x is not observable. Therefore, we need to find an observable variable that can stand in for x in our empirical analysis. It turns out that GDP per capita growth has all the necessary properties.

The approximation $1/x \approx 0$ justifies setting $\theta\phi/\mu\varpi x \approx 0$ in equation (20). This, essentially, says that the ratio of GDP to final production, G/Y, is constant, which is not a bad representation of the data. Then, we write the growth rate

$$g = \zeta\alpha(\mu-1)\varpi x - \rho,$$

and we compute

$$\dot{g} = \zeta\alpha(\mu-1)\varpi\dot{x}$$

and

$$x - x^* = \frac{g+\rho}{\zeta\alpha(\mu-1)\varpi} - \frac{g^*+\rho}{\zeta\alpha(\mu-1)\varpi} = \frac{g-g^*}{\zeta\alpha(\mu-1)\varpi}.$$

Therefore, we can write

$$\dot{g} = \left[\frac{\varphi_2}{\zeta\alpha(\mu-1)\varpi}\left(g-g^*\right) + 2\varphi_2 x^* - \varphi_1\right]\left(g-g^*\right),$$

which we rewrite in the compact form

$$\dot{g} = \left[\xi_1\left(g-g^*\right) + \xi_0\right]\left(g-g^*\right),$$

where:

$$\xi_1 \equiv \frac{\varphi_2}{\zeta\alpha(\mu-1)\varpi}; \tag{48}$$

$$\xi_0 \equiv 2\varphi_2 x^* - \varphi_1. \tag{49}$$

This representation highlights the mapping between coefficients of the quadratic law of motion of the growth rate and the deep parameters of the model.

To take this structure to the data, we work with the differential equation in the form

$$\dot{g} = \left[\xi_1\left(g-g^*\right) + \xi_0\right]\left(g-g^*\right) = ag^2 + bg + c, \tag{50}$$

where $\xi_1 = a$, $\xi_0 - 2\xi_1 g^* = b$ and $\xi_1\left(g^*\right)^2 - \xi_0 g^* = c$. Moreover, since we have again a Riccati equation, we obtain the explicit solution

$$g(t) = \frac{g^*\left(\tilde{g}-g_0\right) - \tilde{g}\left(g^*-g_0\right)e^{a(g^*-\tilde{g})t}}{\left(\tilde{g}-g_0\right) - \left(g^*-g_0\right)e^{a(g^*-\tilde{g})t}}, \tag{51}$$

where g^* is the stable root, $\tilde{g}$ is the unstable root, and g_0 is the initial condition. To interpret properly this solution, which treats the growth rate as a state variable, recall that this construction uses the relation $g = \zeta\alpha(\mu-1)\varpi x - \rho$, which says that in equilibrium the growth rate is a linear function of the state variable x.[12] This solution is a remarkable result

[12] In the language of state-space models, one can think of this linear relation as the measurement equation for a latent variable with known law of motion.

on its own. Furthermore, it says that for any initial condition g_0 in the region of convergence we can study analytically the limiting behavior of the system. In particular, because $g^* > \tilde{g}$ when $\kappa < 1$, which implies $a < 0$, and $g^* < \tilde{g}$ when $\kappa > 1$, which implies $a > 0$, it is always the case for any value of $\kappa > 0$ we obtain

$$\lim_{t \to \infty} g(t) = \frac{g^* \left(\tilde{g} - g_0\right)}{\left(\tilde{g} - g_0\right)} = g^*.$$

This is another way of stating our result that in this model endogenous growth is super robust: a stable steady-state growth rate g^* exists for any positive value of the parameter κ that regulates the curvature of the production of final output, Y, with respect to firm knowledge, Z, which is the model's engine-of-growth factor.

To implement this structure empirically, we employ statistical techniques to estimate the coefficients of (50) and solve for the parameters of interest. Specifically, using a hat on top of a parameter to denote its estimate, we obtain:

$$\hat{\xi}_1 = \hat{a} \Rightarrow \hat{\xi}_1 = \frac{\kappa - 1}{\kappa} \frac{\alpha\zeta(\mu - 1)\varpi}{\zeta\alpha(\mu - 1)\varpi} \Rightarrow \hat{\xi}_1 = \frac{\kappa - 1}{\kappa} \Rightarrow \hat{\kappa} = \frac{1}{1 - \hat{\xi}_1} = \frac{1}{1 - \hat{a}};$$

$$\hat{\xi}_0 = \hat{b} + 2\hat{a}g^*;$$

$$\hat{a}\left(g^*\right)^2 - \left(\hat{b} + 2\hat{a}g^*\right) g^* = \hat{c} \Rightarrow \hat{a}\left(g^*\right)^2 + \hat{b}g^* + \hat{c} = 0.$$

Note that the first equation says that the estimated coefficient of the quadratic equation identifies our key parameter κ. Solving the last two equations, we obtain:

$$\hat{g}^* = \frac{-\hat{b} \pm \sqrt{\hat{b}^2 - 4\hat{a}\hat{c}}}{2\hat{a}};$$

$$\hat{\xi}_0 = \pm\sqrt{\hat{b}^2 - 4\hat{a}\hat{c}}.$$

The first of these two equations estimates the steady state growth rate from the coefficients of the equation that governs the convergence dynamics, and therefore maps it precisely to the deep parameters of the model accounting for such dynamics. Another way to say this is that we do not impose on our transition a value g^* calculated according to some external criterion, but obtain it from the transition itself. In our empirical analysis, we select the root for the stable steady state.

5.3 Speed of Convergence

We now study how the speed of convergence depends on the parameter κ. Focusing on the speed of convergence in an endogenous growth framework is important for two reasons: first, any change that has temporary effects on growth — such as a temporary parameter change or a permanent change in a parameter that affects growth only temporarily (e.g., population size) — has permanent effects on productivity levels. The size of these effects depends on how quickly the system reaches the steady state. All else constant, fast convergence implies small level effects, while slow convergence amplifies them.

The second reason follows from the property that this model delivers a non-constant speed of convergence. A non-constant speed of convergence implies that the magnitude of parameter changes have non-proportional level effects. Furthermore and more importantly, if the speed of convergence differs above and below the steady state, symmetric shocks affect the average growth across shocks, causing it to differ from the steady state growth rate. We explore this implication in the next subsection, here we define and characterize analytically the speed of convergence.

We follow the literature (e.g., Barro and Sala-i-Martin 2004) and use the differential equation

$$\dot{x} = \left[\varphi_2 \left(x - x^*\right) + 2\varphi_2 x^* - \varphi_1\right] \left(x - x^*\right)$$

derived earlier to compute

$$S.o.C. = -\frac{\dot{x}}{x - x^*} = -\left[\varphi_2 \left(x - x^*\right) + 2\varphi_2 x^* - \varphi_1\right]. \tag{52}$$

If we disregard the term $\varphi_2 \left(x - x^*\right)$, we have exactly the formula that is common in the conditional convergence literature, which is simply the linear approximation obtained through a first-order Taylor expansion of a model's convergence equation. Our expression, instead, includes a term that depends on the distance of the state variable from its steady state value. This term is important for the reason stated above, namely that it produces a non-constant and asymmetric speed of convergence. The following proposition summarizes the role of κ for this property.

Proposition 2 *The speed of convergence defined in (52):*

- *increases in the state variable x for $\kappa < 1$;*
- *is independent of the state variable x for $\kappa = 1$;*
- *decreases in the state variable x for $\kappa > 1$.*

Proving this proposition is straightforward. Equation (52) shows that the relationship between the speed of convergence and x depends on the parameter φ_2. From equation (30), the parameter's sign depends exclusively on whether κ is greater or smaller than 1, while the term disappears when $\kappa = 1$. The phase diagrams in Figure 2 then show that for $\kappa = 1$ the change in x is proportional to its level. The role of the non-linearity is visible in the other two cases. For $\kappa < 1$ convergence is faster the further x is above its steady state value. For $\kappa > 1$, instead, convergence speeds up the further x is below its steady state value.

Why is the speed of convergence increasing or decreasing in x depending on the value of κ? To answer this question, it is useful to look at equation (21), which says that the return to vertical innovation depends positively and linearly on x. However, x depends positively or negatively on Z depending on whether $\kappa > 1$ or $\kappa < 1$. Therefore, for $\kappa > 1$, as Z increases, the return to vertical innovation increases, attracting resources away from horizontal innovation. Because entry is the process that ultimately brings the economy to the steady state, reducing investment in entry slows down the convergence process. The reverse happens for $\kappa < 1$: as Z increases, the return to vertical innovation falls and resources flow to horizontal innovation, raising the rate of entry and thus accelerating the convergence process. This mechanism explains why, if Z is particularly large, producing a value of x to the right of the unstable equilibrium $\tilde{x}$ in the phase diagram, the economy explodes because entry is never enough to offset the convexity of the production technology with respect to firm knowledge. None of these effects are present when $\kappa = 1$, in which case x is independent of Z, and convergence occurs at a constant rate.

5.4 Long-Run Growth vs. Steady-State Growth

We now explore an important implication of the non-linear convergence dynamics: symmetric shocks do not even out. Consequently, the time average of growth across shocks with opposite signs differs from the steady state growth rate. This result has important implications for growth theory, namely, because of the asymmetric convergence dynamics, symmetric shocks are a determinant of long-run growth. It follows that focusing on the steady state is important but insufficient to understand growth. Instead, an analyst with that goal has to take into account shocks and transitional dynamics as well.

To illustrate this point in our deterministic model, we focus on surprise scale shocks, i.e., shocks to the scale factor $\Lambda^{\gamma}\Omega^{1-\gamma}$ that are unforeseen by all agents. The reason why this type of shock serves our purpose is twofold. First, as this model has a scale invariant steady state, a permanent change in the scale factor affects only temporarily the model's variables, ensuring that they eventually converge back to steady state. Therefore, we can construct impulse responses without the need to specify a stationary law of motion for the scale factor. Second, the variable x, and consequently g, is linear in the scale factor, implying that a negative and a positive shock of the same magnitude have the same direct effect. Thus, any difference in dynamics following a positive or a negative shock is entirely due to the endogenous propagation mechanism of the model. While this is the simplest shock with these properties that we can consider in this model, our argument generalizes to any stationary shock to any parameter or, we conjecture, to a stochastic version of this model featuring mean-reverting shocks. These alternatives, however, introduce mathematical complications and additional effects that distract from the key insight that we pursue here.

To make our point, we use the concept of average growth across shocks. We consider the symmetric scale shocks defined above and recall that they generate transitional dynamics with growth accelerations/decelerations that eventually die out. We denote the initial effect of these shocks $g_0^+ = g^* + \Delta$ and $g_0^- = g^* - \Delta$. More precisely, these ere the initial jumps in the growth rate caused by, respectively, the permanent positive and negative changes in the scale factor. We then leverage the analytical solution (51) derived in Section 5, interpreting it as the impulse-response function $g(t)$ generated by the shock. We postulate that a positive and a negative shock occur with equal probability 1/2. We then define the average across symmetric shocks

$$\bar{g}(t) = \frac{1}{2}g^+(t) + \frac{1}{2}g^-(t).$$

Our key result is that for $\kappa \neq 1$ average growth across symmetric shocks differs from steady-state growth. The sign of the inequality depends on whether returns to the growth driving factor are increasing or decreasing. The following proposition, proved in the appendix, formalizes this result.

Proposition 3 *Consider symmetric and unanticipated scale shocks with opposite sign and equal probability of occurrence. At any time* $t > 0$ *after the shock, the endogenous propagation mechanism yields:*

- $\overline{g}(t) < g^*$ *for* $\kappa < 1$*;*
- $\overline{g}(t) = g^*$ *for* $\kappa = 1$*;*
- $\overline{g}(t) > g^*$ *for* $\kappa > 1$.

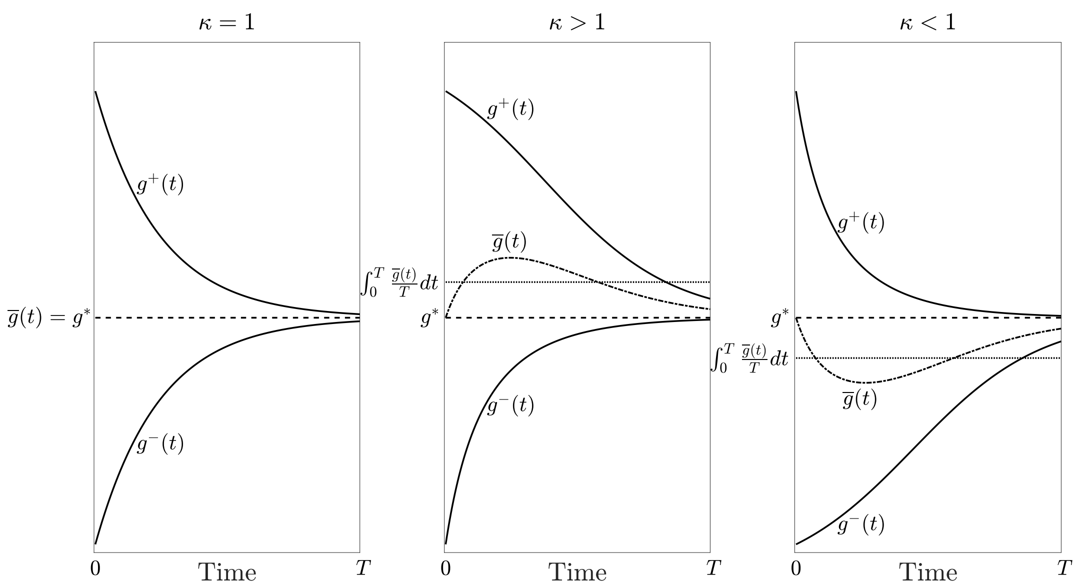


Figure 3: Convergence after Shocks and Steady State vs. Average Growth

Figure 3 provides a graphical illustration of this property and is helpful in developing the intuition for the three cases of constant, increasing, and decreasing returns. The figure depicts on the same time axis the growth dynamics following a positive and a negative shock. It then shows the time path of the average across shocks, $\overline{g}(t)$. At time 0, $\overline{g}(0) = g^*$ in all cases because we have assumed symmetric shocks. Then, $\overline{g}(t)$ coincides at all times with g^* when $\kappa = 1$, meaning that the propagation of the two shocks is symmetric. Instead, the situation changes when non-linearities in the returns to vertical knowledge are present. $\overline{g}(t)$ rises above g^* as time goes by when $\kappa > 1$ and falls below g^* when $\kappa < 1$. It then eventually converges back to g^* as the shock's effect dies out. Finally, the time average of $\overline{g}(t)$ from time $t = 0$ to an arbitrary future period $t = T$ is always above or below g^*. This is an important observation because it implies that computing the time average of the growth rate in an economy with non-constant returns characterized by a succession of this type of shocks does not recover the steady state growth rate. In other words, the frequency and magnitude of these shocks are a determinant of long-run growth.

6 Estimation: The Growth Process in a Quadratic Equation

In this section we estimate the quadratic differential equation in g derived in Section 5. To keep the number of observations high, we work with a panel of countries. The conditions we adopt to select the countries are data availability for at least 100 years, and an expectation that their long-run growth throughout the sample period is primarily driven by technological progress as opposed to the accumulation of some other factor. Based on Maddison data, we find 21 countries that fit the description: USA, France, Italy, Argentina, Sweden, Netherlands, Japan, Canada, Finland, Australia, Norway, Portugal, Switzerland, Uruguay, Great Britain, Spain, Belgium, Germany, Austria, Greece, New Zealand. We use data from 1870 for most countries, but because of data availability issues the first observation for Canada, New Zealand, Austria, and Uruguay is in 1872, while for Argentina and Japan it is in 1902. Admittedly, whether Uruguay belongs to this group is debatable. We note that the results are robust to its exclusion, but we decided to include it as the country-specific parameter coming from its fixed effects term facilitates the interpretation of the results.

Formally, the statistical model is a panel with fixed effects which takes the following form:

$$g_{i,t} - g_{i,t-1} = \varkappa_{1,i} + \varkappa_2 g_{i,t-1} + \varkappa_3 g_{i,t-1}^2 + \epsilon_{i,t}, \tag{53}$$

where ϵ_t is the error term, and the subscript i denotes the country. We estimate the equation by OLS.[13] From this equation, we identify the parameter $\hat{\kappa}$ as illustrated below. Furthermore, by imposing $g_{i,t} - g_{i,t-1} = 0$ we compute the steady state growth rate implied by the statistical model after solving for g_i the quadratic equation. Because the quadratic equation has two solutions, the relevant value of g^* is the stable one. That is the lower if $\kappa > 1$, or the higher if $\kappa < 1$.

Before proceeding with the estimation, whose results we summarize in Table 1, we filter the data to remove high-frequency fluctuations which could bias our speed of convergence. Specifically, we apply the filter on the log-level of income per capita, taking log differences afterwards to retrieve g. Our filter of choice is the HP filter, which is standard in the literature. However, as distinguishing between trend and fluctuations is a controversial issue, we show results after applying the Hamilton filter (Hamilton, 2018) in Table 3, and without filtering in Table 4. The main focus of this paper is the sign of the coefficient $\varkappa_3$, which we show to be robust to different filtering procedures. As expected, the coefficient $\varkappa_2$ is sensitive to the filter.

In our view, the estimate produced from the HP-filtered data are the most credible. The Hamilton filter, instead, delivers results very similar to the ones produced from unfiltered data. To corroborate this view, we compute the speed of convergence from a first-order Taylor approximation of the quadratic equation. The estimated coefficients produce a speed

[13] We have also estimated it by GMM using the estimator proposed by Blundell and Bond (2000). The estimate of $\varkappa_3$, the focus of our empirical exercise, is statistically significant and has the same sign as the OLS estimate. Therefore, our conclusions remain unaltered. The other coefficients are not statistically significant. Nevertheless, the confidence intervals of the OLS coefficients are entirely included within the confidence intervals of the GMM coefficients, implying that we do not detect any issue of consistency of the OLS estimate.

Table 1: Estimation of equation (53) with HP-filtered income per capita log-level.

Variables	Coefficient	Rob.Std.Err	t-stat	p-value
g	-0.030401	0.001824	-16.6718	0.000***
g_squared	0.318743	0.027081	11.7698	0.000***

N = 3044 n = 21 T = 118, . . . , 144.95, . . . , 148 (Unbalaced panel)
R-squared = 0.01188 Adj R-squared = 0.00468
Wald $F(2, 20) = 142.542857$ p-value = 0.0000
RSS = 0.032694 ESS = 0.000417 TSS = 0.033112
Standard errors robust adjusted for 21 clusters.

of convergence of approximately 2% annually for all countries, which is in line with the estimates in the literature, for example Sala-i-Martin (1996).

The coefficient associated with the quadratic term, $\varkappa_3$ is positive, implying that the speed of convergence increases in GDP per capita growth. From equations (30) and (48), we derive a direct mapping from this coefficient to the return to quality $\hat{\kappa} = 1/(1 - \hat{\varkappa}_3) = 1.47$, with 95% confidence interval computed using the delta method of [1.356, 1.585]. We conclude that data from 21 countries in the post industrial revolution period select the growth dynamics delivered by a growth model with increasing returns to firms' technological knowledge.

The steady state income per capita growth rates computed for each country are not statistically different from each other, as $\hat{\varkappa}_1$ exhibits high standard errors. However, the point estimates presented in Table 2 seem reasonable. Countries with a steady state growth rate higher than the US are Scandinavian countries and Canada. Lower growth rates are in continental Europe, with particularly low levels in Southern Europe, Argentina, and Oceania. The country that looks like an outlier is Uruguay, whose steady state growth rate is particularly high. We discuss this case later. Nevertheless, excluding it from the panel does not alter the results. As a further point, we note that the unstable root for g^* is around 8% for all countries.

Table 2 also shows the average growth rate over the sample period per country. As predicted by the theory with $\kappa > 1$, average growth is above steady state growth for all countries except one. The large confidence intervals for $\varkappa_1$ suggests caution in interpreting too much out of the estimates for the steady state growth rates. Nevertheless, the difference between $\overline{g}$ and $\hat{g^*}$ signals that the effect presented in Section 5 is potentially quite large for most countries. This is a direct consequence of a value of κ so far away from 1. The one exception is again Uruguay. A way to rationalize the results for Uruguay is that the country's growth over the sample period may have been characterized to a large extent by the accumulation of a factor that exhibits diminishing returns, such as physical capital. As a result, imposing returns to the growth driving factor estimated also from other 20 countries imposes a value of increasing returns that is far from the actual value for the country. Because g^* depends positively on κ, exaggerating κ while keeping all else constant leads to an upward bias in the value of g^*.

Table 2: Estimated steady state growth rates.

Country	USA	FRA	ITA	ARG	SWE	NDL	JPN
$\hat{g^*}$	1.52%	1.16%	1.04%	0.77%	1.57%	1.13%	1.10%
$mean(g(t))$	1.63%	1.68%	1.71%	1.17%	2.09%	1.61%	2.47%
Country	CAN	FIN	AUS	NOR	PRT	CHE	URY
$\hat{g^*}$	1.68%	1.46%	0.70%	1.70%	1.61%	1.49%	2.21%
$mean(g(t))$	1.90%	2.06%	1.53%	2.49%	1.92%	2.03%	1.12%
Country	GBR	ESP	BEL	DEU	AUT	GRC	NZL
$\hat{g^*}$	1.11%	0.63%	0.96%	1.20%	1.04%	0.20%	0.75%
$mean(g(t))$	1.28%	1.87%	1.51%	1.85%	1.79%	1.65%	1.28%

Table 3: Estimation of equation (53) with Hamilton-filtered income per capita log-level.

Variables	Coefficient	Rob.Std.Err	t-stat	p-value
g	-0.951643	0.026632	-35.7337	0.000***
g_squared	0.259256	0.064072	4.0463	0.001***

N = 3018   n = 21   T = 118, . . . , 143.71, . . . , 148 (Unbalaced panel)
R-squared = 0.49518   Adj R-squared = 0.49147
Wald $F(2, 20)$ = 895.688623   p-value = 0.0000
RSS = 11.623846   ESS = 11.402717   TSS = 23.026563
Standard errors robust adjusted for 21 clusters.

Table 4: Estimation of equation (53) with unfiltered income per capita.

Variables	Coefficient	Rob.Std.Err	t-stat	p-value
g	-0.860514	0.049255	-17.4705	0.000***
g_squared	0.207041	0.089627	2.3100	0.032**

N = 3044   n = 21   T = 118, . . . , 144.95, . . . , 148 (Unbalaced panel)
R-squared = 0.44411   Adj R-squared = 0.44006
Wald $F(2, 20)$ = 405.609198   p-value = 0.0000
RSS = 9.868506   ESS = 7.884559   TSS = 17.753064
Standard errors robust adjusted for 21 clusters.

7 Conclusion

This paper has presented a model that delivers endogenous growth with decreasing, constant, or increasing returns to scale to the growth-driving factor. The model ingredient that achieves this result is the introduction of endogenous competitive forces that affect the incentive to invest. In our model, vertical innovation increases firm-level technological knowledge, the growth driving factor, while horizontal innovation driven by a free-entry condition dilutes the typical firm's market share, affecting the incentives to conduct vertical innovation. As the typical firm's value reflects the non-linear returns to firm knowledge, entry responds to the evolution of such evaluation and causes a market share effect that offsets the non-linear returns to firm knowledge. The result is constant returns to investment that determine the existence of a unique, stable steady state exhibiting endogenous growth.

An important implication of non-linear returns to firm knowledge is that the speed of convergence is not constant and asymmetric. Specifically, convergence is faster from above when returns to firm knowledge are decreasing and faster from below when they are decreasing. This asymmetry implies that symmetric shocks that cause growth accelerations or decelerations that eventually die out introduce an inequality between average growth across shocks and steady state growth, with the sign of this inequality depending exclusively on whether returns to firm knowledge are increasing or decreasing. Therefore, focusing on the steady state is insufficient to understand long-run growth and one needs to take into account the convergence dynamics as well. This suggests that the field can benefit from devoting more attention to the study of the transitional dynamics of growth models.

Finally, the paper estimates the key coefficient that determines the returns to firm knowledge from a panel of 21 countries. It does so by estimating a quadratic differential equation in GDP per capita growth, which is the representation of the law of motion of the key state variable that the model produces under a mild approximation. We find evidence of aggregate increasing returns to firm knowledge.

This paper's results are relevant to the well-known debate concerning the measurement of economic growth (Martin and Riley, 2024). While potential future revisions in growth measurement could result in the rejection of some models that are now considered reliable, the model proposed here is robust as these changes would only affect its parametrization, not its qualitative properties, in particular its ability to deliver endogenous growth.

Appendix: Derivations and Proof

Solution of the Riccati Equations

The generic quadratic differential equation $\dot{x} = ax^2 - bx + c$ has two distinct real roots under the condition $b^2 > 4ac$. We define the roots

$$r_1 = \frac{b - \sqrt{b^2 - 4ac}}{2a} \quad and \quad r_2 = \frac{b + \sqrt{b^2 - 4ac}}{2a}.$$

As discussed in the main body of the paper, the stable steady state is the smaller of the two solutions for $\kappa > 1$ and the larger for $\kappa < 1$. Furthermore, we have $a > 0$ for $\kappa > 1$ and $a < 0$ for $\kappa < 1$. Therefore, in both cases r_1 is the stable root and r_2 is the unstable root. In the main body of the paper, we denote these roots $r_1 = x^*$ and $r_2 = \tilde{x}$ when we solve equation (46) and we denote them $r_1 = g^*$ and $r_2 = \tilde{g}$ when we solve equation (50). For brevity, we solve only equation (46), the procedure is identical for equation (50).

We first factorize the equation

$$ax^2 - bx + c = a(x - r_1)(x - r_2) = a(x - x^*)(x - \tilde{x})$$

so that we can separate the variables, obtaining

$$\frac{dx(t)}{a(x(t) - x^*)(x(t) - \tilde{x})} = dt.$$

Using partial fraction decomposition, we write

$$\frac{1}{a(x(t) - x^*)(x(t) - \tilde{x})} = \frac{A}{x(t) - x^*} + \frac{B}{x(t) - \tilde{x}}.$$

Solving for A and B, we obtain

$$\frac{1}{a(x(t) - x^*)(x(t) - \tilde{x})} = \frac{Aa\left(x(t) - \tilde{x}\right) + Ba\left(x(t) - x^*\right)}{a\left(x(t) - x^*\right)\left(x(t) - \tilde{x}\right)}.$$

$$Aa\left(x(t) - \tilde{x}\right) + Ba\left(x(t) - x^*\right) = 1$$

$$\begin{aligned} Aa\left(x^* - \tilde{x}\right) + Ba\left(x^* - x^*\right) = 1 \quad &and \quad Aa\left(\tilde{x} - \tilde{x}\right) + Ba\left(\tilde{x} - x^*\right) = 1 \\ Aa\left(x^* - \tilde{x}\right) = 1 \quad &and \quad Ba\left(\tilde{x} - x^*\right) = 1 \\ A = \frac{1}{a(x^* - \tilde{x})} \quad &and \quad B = \frac{-1}{a(x^* - \tilde{x})}. \end{aligned}$$

$$\begin{aligned} \frac{Aa\left(x(t) - \tilde{x}\right) + Ba\left(x(t) - x^*\right)}{a\left(x(t) - x^*\right)\left(x(t) - \tilde{x}\right)} &= \frac{\frac{1}{a(x^* - \tilde{x})} a\left(x(t) - \tilde{x}\right) - \frac{1}{a(x^* - \tilde{x})} a\left(x(t) - x^*\right)}{a\left(x(t) - x^*\right)\left(x(t) - \tilde{x}\right)} \\ &= \frac{1}{(x^* - \tilde{x})} \frac{\left(x(t) - \tilde{x}\right) - \left(x(t) - x^*\right)}{a\left(x(t) - x^*\right)\left(x(t) - \tilde{x}\right)} \\ &= \frac{1}{a\left(x(t) - x^*\right)\left(x(t) - \tilde{x}\right)}. \end{aligned}$$

Thus, the integral simplifies to

$$\int \left(\frac{A}{x(t) - x^*} + \frac{B}{x(t) - \tilde{x}} \right) dx = \int dt,$$

which yields

$$\frac{1}{a(x^* - \tilde{x})} \ln|x(t) - x^*| - \frac{1}{a(x^* - \tilde{x})} \ln|x(t) - \tilde{x}| = t + C.$$

Rearranging terms, we have

$$\ln\left|\frac{x(t) - x^*}{x(t) - \tilde{x}}\right| = a(x^* - \tilde{x})t + C.$$

Solving this expression for $x(t)$, we obtain

$$\frac{x(t)-x^*}{x(t)-\tilde{x}} = e^{a(x^*-\tilde{x})t+C} = e^{a(x^*-\tilde{x})t+C} = Ce^{a(x^*-\tilde{x})t}.$$

Hence,

$$\begin{aligned}
\frac{x(t)-x^*}{x(t)-\tilde{x}} &= Ce^{a(x^*-\tilde{x})t} \\
x(t) - Ce^{a(x^*-\tilde{x})t}x(t) &= x^* - Ce^{a(x^*-\tilde{x})t}\tilde{x} \\
x(t) &= \frac{x^* - \tilde{x}Ce^{a(x^*-\tilde{x})t}}{1-Ce^{a(x^*-\tilde{x})t}}.
\end{aligned}$$

We can then solve for C as follows

$$x(0) = \frac{x^*-C\tilde{x}}{1-C} \Rightarrow C = \frac{x^*-x(0)}{\tilde{x}-x(0)}.$$

Thus, our final solution is

$$x(t) = \frac{x^*\left(\tilde{x}-x(0)\right) - \tilde{x}\left(x^*-x(0)\right)e^{a(x^*-\tilde{x})t}}{\left(\tilde{x}-x(0)\right) - \left(x^*-x(0)\right)e^{a(x^*-\tilde{x})t}},$$

where by construction $a = \varphi_2$.

Proof of Proposition 3

We want to study the difference

$$\bar{g}(t) - g^* = \frac{1}{2}g^+(t) + \frac{1}{2}g^-(t) - g^* \lesseqgtr 0.$$

We write it

$$\bar{g}(t) - g^* = \frac{1}{2}\left[g^+(t) - g^*\right] + \frac{1}{2}\left[g^-(t) - g^*\right] \lesseqgtr 0.$$

We can then do the brute force algebra:

$$\begin{aligned}
\frac{(g^*-\tilde{g})(g^*-g_0^+)e^{-a(\tilde{g}-g^*)t}}{(\tilde{g}-g_0^+)+(g^*-g_0^+)e^{-a(\tilde{g}-g^*)t}} + \frac{(g^*-\tilde{g})(g^*-g_0^-)e^{-a(\tilde{g}-g^*)t}}{(\tilde{g}-g_0^-)+(g^*-g_0^-)e^{-a(\tilde{g}-g^*)t}} &\lesseqgtr 0 \\
(\tilde{g}-g^*)\,\Delta e^{-a(\tilde{g}-g^*)t}\left[\frac{1}{(\tilde{g}-g_0^+)+\Delta e^{-a(\tilde{g}-g^*)t}} - \frac{1}{(\tilde{g}-g_0^-)-\Delta e^{-a(\tilde{g}-g^*)t}}\right] &\lesseqgtr 0.
\end{aligned}$$

The sign of the term outside of the square bracket depends on κ. Specifically, when $\kappa > 1$, $a > 0$, thus g^* is the lower root and $\tilde{g}$ is the higher root. The opposite is true when $\kappa < 1$, whereas when $\kappa = 1$, $g^* = \tilde{g}$. Consequently, the sign of the term outside of brackets is positive for $\kappa > 1$, negative for $\kappa < 1$, and equal to 0 for $\kappa = 1$. Note that this last statement proves the second bullet point in the proposition. Therefore, the remainder of the proof focuses on the cases where $\kappa \neq 1$.

To end the proof, we therefore need to prove that the term within the parenthesis is positive for $t > 0$ when $\kappa \neq 1$. That happens when

$$g_0^+ - g_0^- > 2\Delta e^{-a(\tilde{g}-g^*)t}.$$

Because $g_0^+ - g_0^- = 2\Delta$, the statement is true for

$$e^{-a(\tilde{g}-g^*)t} < 1.$$

Taking natural logs on both sides, the expression becomes

$$-a(\tilde{g} - g^*)t < 0.$$

The statement is always true for $\kappa \neq 1$ because $\tilde{g} > g^*$ when $a > 0$ (thus $\kappa > 1$), while $\tilde{g} < g^*$ when $a < 0$ (thus $\kappa < 1$).

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